SINGULAR SELF-ADJOINT BOUNDARY VALUE PROBLEMS FOR THE DIFFERENTIAL EQUATION $Lx=\lambda Mx$[1]

BY

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1. **Introduction.** Let L be a linear differential operator. Eigenfunction expansions for self-adjoint boundary value problems associated with the differential equation $Lx=\lambda x$ on an infinite interval have been treated by various authors, beginning with Weyl [4] in 1910. Boundary value problems on a finite interval for differential equations of the form $Lx=\lambda Mx$, where M is another linear differential operator whose order is less than that of L, have been discussed by Kamke [3]. The purpose of the first part of this paper is to improve the results of Kamke and to state them in a form suitable for the study of boundary value problems on an infinite interval. In the remainder of the paper the methods used by Coddington and Levinson [1] for the equation $Lx=\lambda x$ are modified to study the equation $Lx=\lambda Mx$ on an infinite interval.

2. **Self-adjoint boundary value problems on a finite interval.** We consider the problem $Lx=\lambda Mx$, $Ux=0$ on the finite interval I, $a\leqq t\leqq b$. Here $Lx=\sum_{i=0}^{n} p_i(t)x^{(n-i)}$, $Mx=\sum_{i=0}^{m} q_i(t)x^{(m-i)}$, are formally self-adjoint differential operators, with $p_i(t)$ and $q_i(t)$ complex-valued functions with the properties that $p_i(t)$ is of class C^{n-i} on I, $q_i(t)$ is of class C^{m-i} on I, and $p_0(t)$ and $q_0(t)$ do not vanish on I. We also require that $0\leqq m\leqq n-2$. The boundary conditions $Ux=0$ consist of n linearly independent conditions of the form $U_jx=\sum_{k=1}^{n}\left[m_{jk}x^{(k-1)}(a)+n_{jk}x^{(k-1)}(b)\right]=0$ $(j=1, 2, \cdots, n)$. A value of λ for which a nontrivial solution exists is called an eigenvalue and a corresponding nontrivial solution is called an eigenfunction. Let D be the set of functions of class C^n on I which satisfy the boundary conditions $Ux=0$. For any functions u, v belonging to $L^2(I)$ the inner product $(u, v)=\int_I u(t)\bar{v}(t)dt$ is defined. We assume the problem self-adjoint in the sense that $(Lu, v)=(u, Lv)$ and $(Mu, v)=(u, Mv)$ for all $u, v\in D$. We assume also that there exist real constants $d>0$ and K such that $(Mu, u)\geqq d(u, u)$ and $(Lu, u)\geqq K(u, u)$ for all $u\in D$.

LEMMA 2.1. *Either every complex number λ is an eigenvalue for the boundary value problem $Lx=\lambda Mx$, $Ux=0$ or there are at most countably many eigenvalues with no finite limit point.*

Presented to the Society, December 27, 1956 under the title *Eigenfunction expansions for a class of singular boundary value problems*; received by the editors December 10, 1956.

(1) This paper is based on the author's doctoral dissertation, submitted to the Massachusetts Institute of Technology in June 1956. The author is grateful to Professor N. Levinson for his supervision of this dissertation.

Proof. Choose c in $a<c<b$ and let $\phi_j(t, \lambda)$ $(j=1, \cdots, n)$ be the solution of $Lx=\lambda Mx$ such that $\phi_j^{(k-1)}(c, \lambda)=\delta_{jk}$ $(j, k=1, \cdots, n)$, where δ_{jk} is the Kronecker δ. Then every solution of $Lx=\lambda Mx$ is a linear combination of the $\phi_j(t, \lambda)$. If there are p and no more than p linearly independent eigenfunctions for $\lambda=\lambda_0$, then every eigenfunction for $\lambda=\lambda_0$ can be written as a linear combination of these p eigenfunctions, and it is clear that $1\leqq p\leqq n$. The function $x(t, \lambda_0)$ is an eigenfunction if and only if the coefficients c_j can be chosen so that if $x(t, \lambda_0)=\sum_{j=1}^n c_j\phi_j(t, \lambda_0)$, then $Ux=\sum_{j=1}^n c_jU\phi_j(t, \lambda_0)=0$. This can be done if and only if $\Delta(\lambda_0)=0$, where $\Delta(\lambda)=\det\left[U_i\phi_j(t, \lambda)\right]$. If the matrix of $\Delta(\lambda_0)$ has rank $(n-p)$, then λ_0 is an eigenvalue of order p and there are p linearly independent eigenfunctions for λ_0. Since the functions $\phi_j^{(k-1)}(t, \lambda)$ are entire functions of λ for each fixed t, in particular for $t=a$ and $t=b$, it follows that $\Delta(\lambda)$ is an entire function of λ. Thus either $\Delta(\lambda)\equiv 0$, in which case every complex λ is an eigenvalue, or there are at most countably many eigenvalues with no finite limit point, corresponding to the zeros of $\Delta(\lambda)$.

Next, we consider the nonhomogeneous boundary value problem $Lx=\lambda Mx+f$, $Ux=0$ for any function f continuous on I.

THEOREM 2.1. *If for at least one value of λ the problem $Lx=\lambda Mx$, $Ux=0$ has no solution except the trivial one, then there exists a unique function $G(t, \tau, \lambda)$ defined for $t, \tau\in I$ and all λ except the eigenvalues, and having the following properties:*

(i) $(\partial^kG/\partial t^k)$ $(k=0, 1, \cdots, n-2)$ *exist and are continuous in (t, τ, λ) for $t, \tau\in I$ and λ not an eigenvalue. Also, $\partial^kG/\partial t^k$ for $k=n-1$, n are continuous on $a\leqq t<\tau\leqq b$ and $a\leqq\tau\leqq t\leqq b$, λ not an eigenvalue. For fixed t and τ these functions are meromorphic in λ.*

(ii)
$$\frac{\partial^{n-1}G(\tau+0, \tau, \lambda)}{\partial t^{n-1}}-\frac{\partial^{n-1}G(\tau-0, \tau, \lambda)}{\partial t^{n-1}}=\frac{1}{p_0(\tau)}.$$

(iii) *As a function of t, G satisfies $Lx=\lambda Mx$ if $t\neq\tau$.*

(iv) *As a function of t, G satisfies $Ux=0$ for $a\leqq\tau\leqq b$.*

(v) *The nonhomogeneous problem $Lx=\lambda Mx+f$, $Ux=0$ has the unique solution $\int_a^bG(t, \tau, \lambda)f(\tau)d\tau$, if λ is not an eigenvalue, for any function f continuous on I.*

Proof. We define

$$K(t, \tau, \lambda)=\frac{1}{p_0(\tau)W(\phi_1, \cdots, \phi_n; \tau)}\begin{vmatrix}\phi_1(\tau, \lambda) & \cdots & \phi_n(\tau, \lambda)\\ \vdots & & \vdots\\ \phi_1^{(n-2)}(\tau, \lambda) & \cdots & \phi_n^{(n-2)}(\tau, \lambda)\\ \phi_1(t, \lambda) & \cdots & \phi_n(t, \lambda)\end{vmatrix}\quad\text{for } t\geqq\tau$$

and $K(t, \tau, \lambda)=0$ for $t<\tau$. The Wronskian $W(\phi_1, \cdots, \phi_n; \tau)$ depends only on τ since $W(\phi_1, \cdots, \phi_n; \tau) = \exp - \int_c^\tau (p_1(s)/p_0(s))ds$. Since the determinant in the numerator vanishes when any two rows are identical, it is clear that $(\partial^j K/\partial t^j)(\tau+0, \tau, \lambda)=0$ $(j=0, 1, \cdots, n-2)$, so that $\partial^j K/\partial t^j$ is continuous in (t, τ, λ) for $j=0, 1, \cdots, n-2$; $t, \tau \in I$; and all λ, as well as entire in λ for fixed (t, τ). For $j=n-1$ or n, $\partial^j K/\partial t^j$ is continuous in (t, τ, λ) for all λ and $a \leqq \tau \leqq t \leqq b$ or $a \leqq t < \tau \leqq b$. At $t=\tau$, $\partial^{n-1}K/\partial t^{n-1}$ has a simple discontinuity of magnitude $1/p_0(\tau)$. For $t \neq \tau$, $L_t K = \lambda M_t K$, the subscripts indicating that the differentiations are with respect to t. It follows that $\int_a^b K(t, \tau, \lambda)f(\tau)d\tau$ is of class C^n in t, entire in λ for fixed t, and satisfies $Lx=\lambda Mx+f$. We now modify $K(t, \tau, \lambda)$ so that the boundary conditions $Ux=0$ are also satisfied. We let $G(t, \tau, \lambda)=K(t, \tau, \lambda)+\sum_{j=1}^n c_j\phi_j(t, \lambda)$ and attempt to choose the c_j so that $UG = 0$ for fixed τ. Then for $k = 1, \cdots, n$ we must have $U_kG = U_kK + \sum_{j=1}^n c_j U_k\phi_j=0$. The determinant of this linear system is $\Delta(\lambda)$, which does not vanish unless λ is an eigenvalue. Thus the c_j are determined as functions of (τ, λ) continuous for $\tau \in I$ and all λ which are not eigenvalues, and meromorphic in λ for fixed τ. It is clear that $\int_a^b G(t, \tau, \lambda)f(\tau)d\tau$ is a solution of the nonhomogeneous problem. Since the difference between any two solutions of the nonhomogeneous problem satisfies $Lx=\lambda Mx$, $Ux=0$ and λ is not an eigenvalue, this solution is unique. It remains only to prove that $G(t, \tau, \lambda)$ is unique. If there were two functions G and $\tilde{G}$ with the above properties for some λ which is not an eigenvalue, then as a function of t, $G-\tilde{G}$ would be of class C^{n-1} and would satisfy $Lx=\lambda Mx$ for $t\neq\tau$. Then $G-\tilde{G}$ would be of class C^n. Since λ is not an eigenvalue, $G-\tilde{G}$ would vanish identically, and thus G is unique.

The function $G(t, \tau, \lambda)$ is called the Green's function of the boundary value problem. If $\lambda=0$ is not an eigenvalue, then $G(t, \tau, 0)$ exists. This function will be denoted by $G(t, \tau)$. The unique solution of $Lx=f$, $Ux=0$ is

$$\int_a^b G(t, \tau)f(\tau)d\tau.$$

Two functions $u, v \in D$ will be called orthogonal if $(Mu, v)=(u, Mv)=0$.

THEOREM 2.2. *The boundary value problem $Lx=\lambda Mx$, $Ux=0$ has at most countably many eigenvalues which are all real and have no finite limit point in the λ-plane. If there exist eigenvalues, they can be arranged in a sequence $\lambda_1, \lambda_2, \cdots$ with $\lambda_1 \leqq \lambda_2 \leqq \cdots$. There is a corresponding sequence of eigenfunctions x_j which can be chosen orthonormal in the sense that $(Mx_j, x_k)=\delta_{jk}$. Every eigenfunction belonging to an eigenvalue can be written as a linear combination of the finite number of x_j belonging to this eigenvalue.*

Proof. Let λ_0 be an eigenvalue and x_0 a corresponding eigenfunction. Then $Lx_0 = \lambda_0 Mx_0$ and by the self-adjointness conditions, $(Lx_0, x_0) =\lambda_0(Mx_0, x_0)$, $(x_0, Lx_0)=(x_0, \lambda_0 Mx_0)=\bar{\lambda}_0(x_0, Mx_0)=\bar{\lambda}_0(Mx_0, x_0)$. This implies $(\lambda_0-\bar{\lambda}_0)(Mx_0, x_0)=0$, and since $(Mx_0, x_0)\neq 0$, $\lambda_0=\bar{\lambda}_0$, so that λ_0 is real. Next we show that eigenfunctions corresponding to different eigenvalues are orthogonal. Let x_1 be an eigenfunction with eigenvalue λ_1 and x_2 an eigenfunction with eigenvalue λ_2. Then $0=(Lx_1, x_2)-(x_1, Lx_2)=\lambda_1(Mx_1, x_2)-\lambda_2(x_1, Mx_2)=(\lambda_1-\lambda_2)(Mx_1, x_2)$. Since $\lambda_1\neq\lambda_2$, x_1 and x_2 are orthogonal. There is a finite number of linearly independent eigenfunctions corresponding to any eigenvalue, and these may be replaced by an orthogonal system of eigenfunctions, using a well-known procedure, such that any eigenfunction corresponding to this eigenvalue is a linear combination of these orthogonal eigenfunctions. Since all eigenvalues are real, not all λ are eigenvalues, and Lemma 2.1 may be applied. If λ_0 is any eigenvalue and x_0 a corresponding eigenfunction, $(Lx_0, x_0)=\lambda_0(Mx_0, x_0)$, and since $(Lx_0, x_0)\geqq K(x_0, x_0)$, $(Mx_0, x_0)\geqq d(x_0, x_0)$, it follows that $\lambda_0\geqq K/d$. Thus the eigenvalues are bounded below and can be arranged in an increasing sequence. This completes the proof of the theorem.

To prove the existence of eigenvalues we require some properties of the Green's function. We have already seen that for fixed t, τ, $G(t, \tau, \lambda)$ is a meromorphic function of λ, analytic in any region where $\Delta(\lambda)\neq 0$, and that the zeros of $\Delta(\lambda)$ are isolated.

LEMMA 2.2. *If λ is not an eigenvalue and $t, \tau\in I$, then $G(t, \tau, \bar\lambda)=\bar G(\tau, t, \lambda)$.*

Proof. Let f, g be continuous on I and define $u(t)=\int_a^b G(t, \tau, \bar\lambda)f(\tau)d\tau$, $v(t)=\int_a^b G(t, \tau, \lambda)g(\tau)d\tau$. Then $u, v\in D$ and $Lu-\bar\lambda Mu=f$, $Lv-\lambda Mv=g$, by Theorem 2.1. Now $(f, v)=(Lu-\bar\lambda Mu, v)=(Lu, v)-\bar\lambda(Mu, v)=(u, Lv-\lambda Mv)=(u, g)$. Thus $\int_a^b\int_a^b G(t, \tau, \bar\lambda)f(\tau)\bar g(t)d\tau dt=\int_a^b\int_a^b \bar G(\tau, t, \lambda)f(\tau)\bar g(t)dt d\tau$ for all continuous f, g, and this implies $G(t, \tau, \bar\lambda)=\bar G(\tau, t, \lambda)$.

LEMMA 2.3. *Let λ_0 be a zero of $\Delta(\lambda)$ of any order k. Then $G(t, \tau, \lambda)$ has, at worst, a pole of order one at λ_0.*

Proof. For $t, \tau\in I$ and λ in some region $0<|\lambda-\lambda_0|<\rho$, we can write

$$G(t, \tau, \lambda) = \sum_{i=-p}^{\infty} h_i(t, \tau)(\lambda - \lambda_0)^i,$$

with $h_i(t, \tau)=(1/2\pi i)\int_C(G(t, \tau, \lambda)/(\lambda-\lambda_0)^{i+1})d\lambda$, where C can be taken as the circle $|\lambda-\lambda_0|=\rho_0<\rho$ in the positive sense. We must show that $h_i(t, \tau)\equiv 0$ for $i\leqq -2$. Since $G(t, \tau, \lambda)$ is continuous on C, $|h_i(t, \tau)|\leqq A/\rho^i$ for $t, \tau\in I$ with A independent of i. Thus the series for $G(t, \tau, \lambda)$ converges uniformly in $a\leqq t, \tau\leqq b$ for every fixed λ in some region $0<|\lambda-\lambda_0|\leqq\rho_1$. Let f be an arbitrary continuous function on I and define $H(t, \lambda)=\int_a^b G(t, \tau, \lambda)f(\tau)d\tau$, so that $H(t, \lambda)=\sum_{i=-p}^{\infty} H_i(t)(\lambda-\lambda_0)^i$, with $H_i(t)=\int_a^b h_i(t, \tau)f(\tau)d\tau$. This series converges uniformly on I for every fixed λ in $0<|\lambda-\lambda_0|\leqq\rho_1$. The function $H(t, \lambda)$ is of class C^n in t for such λ and is the solution of $Lx=\lambda Mx+f$, $Ux=0$. Since $H_i(t)=(1/2\pi i)\int_C(H(t, \lambda)/(\lambda-\lambda_0)^{i+1})d\lambda$, the $H_i(t)$ are also of class C^n and the series $H(t, \lambda)=\sum_{i=-p}^{\infty} H_i(t)(\lambda-\lambda_0)^i$ can be differentiated term by term n times. Substituting in $LH=\lambda MH+f$, $UH=0$, we obtain

$$\sum_{i=-p}^{\infty} (\lambda - \lambda_0)^i[LH_i - \{(\lambda - \lambda_0) + \lambda_0\}MH_i] = f, \sum_{i=-p}^{\infty} U_jH_i(\lambda - \lambda_0)^i = 0$$

$$(j = 1, \cdots, n).$$

Let r be the smallest integer such that $H_r(t) \not\equiv 0$, and suppose $r \leqq -2$. Then $LH_r = \lambda_0 MH_r$, $UH_r = 0$, and $LH_{r+1} = \lambda_0 MH_{r+1} + MH_r$, $UH_{r+1} = 0$. Thus $0 = (LH_{r+1}, H_r) - (H_{r+1}, LH_r) = \lambda_0[(MH_{r+1}, H_r) - (H_{r+1}, MH_r)] + (MH_r, H_r)$, and $(MH_r, H_r) = 0$, which implies $H_r \equiv 0$. Now $H_i(t) = \int_a^b h_i(t, \tau) f(\tau) d\tau = 0$ for every continuous f if $i \leqq -2$, and hence $h_i(t, \tau) \equiv 0$ for $i \leqq -2$.

LEMMA 2.4. *If λ_0 is an eigenvalue of order k and $y_1, \cdots, y_k$ is a corresponding set of orthonormal eigenfunctions, then the residue $R(t, \tau)$ of $G(t, \tau, \lambda)$ at λ_0 is $-\sum_{i=1}^k y_i(t)\bar{y}_i(\tau)$.*

Proof. Since $Ly_i - \lambda My_i = (\lambda_0 - \lambda)My_i$, $y_i(t) = (\lambda_0 - \lambda)\int_a^b G(t, \tau, \lambda) My_i(\tau) d\tau$. As $\lambda \to \lambda_0$, $(\lambda_0 - \lambda)G(t, \tau, \lambda) \to -R(t, \tau)$, and thus $y_i(t) = -\int_a^b R(t, \tau) My_i(\tau) d\tau$. Since $G(t, \tau, \lambda) - K(t, \tau, \lambda)$ is of class C^n and $K(t, \tau, \lambda)$ is entire in λ, $R(t, \tau)$ is also the residue of $G(t, \tau, \lambda) - K(t, \tau, \lambda)$ at λ_0. Then $R(t, \tau) = (1/2\pi i) \cdot \int_C [G(t, \tau, \lambda) - K(t, \tau, \lambda)] d\lambda$, where C is a small circle about λ_0. Thus $R(t, \tau)$ is of class C^n in t and τ. Since $LG = \lambda MG$ for $t \neq \tau$ and $UG = 0$, $LG(t, \tau, \lambda) - \lambda_0 MG(t, \tau, \lambda) = (\lambda - \lambda_0)MG(t, \tau, \lambda)$, $UG = 0$ for $t \neq \tau$. Expanding $G(t, \tau, \lambda)$ in a Laurent series about λ_0 and equating coefficients, $L_t R(t, \tau) = \lambda_0 M_t R(t, \tau)$, $U_t R = 0$. Since R is of class C^n in t, this also holds for $t = \tau$, and $R(t, \tau)$ is an eigenfunction at λ_0. Thus $R(t, \tau) = \sum_{i=1}^k a_i(\tau) y_i(t)$, with the $a_i(\tau)$ of class C^n on I for $i = 1, \cdots, k$. Using Lemma 2.2 it follows that $\bar{a}_i(\tau)$ is an eigenfunction at λ_0, and $\bar{a}_i(\tau) = \sum_{p=1}^k \bar{b}_{pi} y_p(\tau)$. It follows from $y_i(t) = -\int_a^b R(t, \tau) My_i(\tau) d\tau$ that $b_{pi} = -\delta_{pi}$ and $\bar{a}_i(\tau) = -y_i(\tau)$, which yields $R(t, \tau) = -\sum_{i=1}^k y_i(t)\bar{y}_i(\tau)$.

THEOREM 2.3. *If a function $f \in D$ is orthogonal to all eigenfunctions x_k, then f is the zero function.*

Proof. For $f \in D$ we define $H(t, \lambda) = \int_a^b G(t, \tau, \lambda) Mf(\tau) d\tau$. Then $H(t, \lambda)$ is of class C^n in t, meromorphic in λ, with at worst simple poles at the zeros of $\Delta(\lambda)$, and $LH = \lambda MH + Mf$, $UH = 0$. At any pole λ_0 of $G(t, \tau, \lambda)$, the residue of $G(t, \tau, \lambda)$ is $-\sum_{i=1}^k y_i(t)\bar{y}_i(\tau)$, where $y_1, \cdots, y_k$ are orthonormal eigenfunctions at λ_0. The residue of $H(t, \lambda)$ at λ_0 is $-\sum_{i=1}^k y_i(t) \int_a^b Mf(\tau)\bar{y}_i(\tau) d\tau$, which vanishes since f is orthogonal to each y_i. Thus $H(t, \lambda)$ is an entire function of λ and may be written as $\sum_{i=0}^\infty a_i(t)\lambda^i$, with each $a_i(t)$ of class C^n. Substitution of this series in $LH = \lambda MH + Mf$, $UH = 0$ gives $La_0 = Mf$, $Ua_0 = 0$; $La_k = Ma_{k-1}$, $Ua_k = 0$, $(k = 1, 2, \cdots)$. We now have $(Ma_{j-1}, a_k) = (La_j, a_k) = (a_j, La_k) = (a_j, Ma_{k-1}) = (Ma_j, a_{k-1})$, and thus $W_{j+k} = (Ma_j, a_k)$ depends only on $j + k$, not on j and k separately. Since H and its first n derivatives in t are entire functions of λ, $h(\lambda) = (MH, a_0)$ is entire in λ, $h(\lambda) = \sum_{i=0}^\infty (Ma_i, a_0)\lambda^i = \sum_{i=0}^\infty W_i\lambda^i$. Also, $2^{-1}[h(\lambda) + h(-\lambda)] = \sum_{i=0}^\infty W_{2i}\lambda^{2i}$ is entire in λ. The Schwarz inequality for the inner product (Mf, g) is $|(Mf, g)|^2$

$\leqq (Mf, f)(Mg, g)$ for all $f, g \in D$. Thus $W_{2i}^2 = (Ma_{i-1}, a_{i+1})^2 \leqq (Ma_{i-1}, a_{i-1}) \cdot (Ma_{i+1}, a_{i+1}) = W_{2i-2}W_{2i+2}$. If $W_2 \neq 0$, then $W_{2i}/W_{2i-2} \leqq W_{2i+2}/W_{2i}$ for $i = 1, 2, \cdots$. This implies that the series $\sum_{i=0}^{\infty} W_{2i}\lambda^{2i}$ has a finite radius of convergence, which is false. Thus $W_2 = 0$. Since $W_2 = (Ma_1, a_1)$, $a_1(t) \equiv 0$, and $Ma_0 = La_1 = 0$, which yields $a_0 \equiv 0$. Now $Mf = La_0 = 0$, and $f \equiv 0$.

Theorem 2.3 implies the existence of eigenvalues for the problem $Lx = \lambda Mx$, $Ux = 0$ as an immediate corollary. On D we have an inner product defined by $[f, g] = (Mf, g) = (f, Mg)$. Let H be the Hilbert space completion of D in this inner product. Then H is a Hilbert space with inner product $[f, g]$ and norm $\|f\| = [f, f]^{1/2}$. For any $f \in H$, we define the Fourier coefficients $a_k = [f, x_k]$ for each eigenfunction x_k.

Lemma 2.5. *There is no loss in generality in assuming $(Lu, u) > 0$ for all $u \neq 0$ in D.*

Proof. We have assumed $(Lu, u) \geqq K(u, u)$, and $(Mu, u) = d(u, u)$ for some real $d > 0$, K, and all $u \in D$. For any $\Lambda > -K/d$, let $L_1u = Lu + \Lambda Mu$. Then $(L_1u, u) = (Lu, u) + \Lambda(Mu, u) > 0$ for all $u \in D$. The problem $L_1x = \lambda Mx$, $Ux = 0$ has the same eigenfunctions x_k as $Lx = \lambda Mx$, $Ux = 0$, and eigenvalues $\lambda_k + \Lambda$. Thus we may use L_1 in place of L without changing the eigenfunctions, and we may assume $(Lu, u) > 0$ from the outset. This assumption will be made in the remainder of this section. We now have $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$.

Lemma 2.6. *If $f \in D$, then $(Lf, f) \geqq \sum_{k=1}^{\infty} \lambda_k |a_k|^2$.*

Proof. For any integer p, $(L(f - \sum_{k=1}^{p} a_k x_k), f - \sum_{k=1}^{p} a_k x_k) \geqq 0$. Thus $(Lf, f) - \sum_{k=1}^{p} a_k(Lx_k, f) - \sum_{k=1}^{p} \bar{a}_k(Lf, x_k) + \sum_{j,k=1}^{p} a_j\bar{a}_k(Lx_j, x_k) \geqq 0$, or $(Lf, f) - \sum_{k=1}^{p} \lambda_k a_k(Mx_k, f) - \sum_{k=1}^{p} \lambda_k a_k(f, Mx_k) + \sum_{j,k=1}^{p} \lambda_j a_j \bar{a}_k(Mx_j, x_k) \geqq 0$. Using the orthogonality $(Mx_j, x_k) = \delta_{jk}$ and the definition of a_k, we obtain $(Lf, f) - \sum_{k=1}^{p} \lambda_k |a_k|^2 \geqq 0$. Since this holds for all p, $\sum_{k=1}^{\infty} \lambda_k |a_k|^2$ converges and $(Lf, f) \geqq \sum_{k=1}^{\infty} \lambda_k |a_k|^2$.

We now examine the convergence of the expansion $\sum_{k=1}^{\infty} a_k x_k$ to the function f. It is easy to see that the partial derivatives $G^{(p,q)}(t, \tau) = (\partial^p/\partial t^p) \cdot (\partial^q/\partial \tau^q)G(t, \tau)$ exist for $p, q \leqq n-1$ if $t \neq \tau$ and are continuous for $t \neq \tau$. Also, $G^{(p,q)}(\tau + 0, \tau)$ is continuous for $\tau \in I$.

Lemma 2.7. *The series $\sum_{k=1}^{\infty} |x_k^{(j)}(t)|^2/\lambda_k$ converges for $j \leqq n-1$ and its sum is no greater than $G^{(j,j)}(t+0, t)$.*

Proof. Let $w(t)$ be any function of class C^n which is not identically zero and which vanishes together with its first n derivatives at a and b. The problem $Lv = w^{(j)}$, $Uv = 0$ $(j = 0, 1, \cdots, n-1)$ has a unique solution $v_j(t) = \int_a^b G(t, \tau)w^{(j)}(\tau)d\tau$. Let v_j have Fourier coefficients a_k^j. Then Lemma 2.6 gives $\sum_{k=1}^{p} \lambda_k |a_k^j|^2 \leqq (Lv_j, v_j)$. Integrating by parts and using the boundary conditions on w, $(Lv_j, v_j) = (w^{(j)}, v_j) = (-1)^j(w, v_j^{(j)})$. But integration by parts applied to $v_j^{(j)}(t)=\int_a^b G^{(j,0)}(t,\tau)w^{(j)}(\tau)d\tau$ yields

$$v_j^{(j)}(t) = (-1)^j \int_a^b G^{(j,j)}(t,\tau)w(\tau)d\tau.$$

It follows that

$$(Lv_j, v_j) = \left(w, \int_a^b G^{(j,j)}(t,\tau)w(\tau)d\tau\right) = \int_a^b\int_a^b \overline{G}^{(j,j)}(t,\tau)w(t)\bar{w}(\tau)dtd\tau.$$

Also, $a_k^j=(v_j, Mx_k)=(v_j, Lx_k)/\lambda_k=(Lv_j, x_k)/\lambda_k=(w^{(j)}, x_k)/\lambda_k$, and integration by parts gives $a_k^j=(-1)^j(w, x_k^{(j)})/\lambda_k$. Thus, for any p, $\sum_{k=1}^p \lambda_k|a_k^j|^2 \leqq \sum_{k=1}^p \int_a^b\int_a^b \bar{x}_k^{(j)}(t)x_k^{(j)}(\tau)w(t)\bar{w}(\tau)dtd\tau/\lambda_k$. Now, substitution in the Bessel inequality $\sum_{k=1}^p \lambda_k|a_k^j|^2 \leqq (Lv_j, v_j)$ gives

$$\int_a^b\int_a^b \left[\overline{G}^{(j,j)}(t,\tau) - \sum_{k=1}^p \bar{x}_k^{(j)}(t)x_k^{(j)}(\tau)/\lambda_k\right] w(t)\bar{w}(\tau)dtd\tau \geqq 0.$$

Since this holds for arbitrary functions w of class C^n which vanish identically outside of a closed subinterval of I, $Rl[\overline{G}^{(j,j)}(t,\tau)-\sum_{k=1}^p \bar{x}_k^{(j)}(t)x_k^{(j)}(\tau)/\lambda_k] \geqq 0$, taking real parts. Letting $\tau\to t$, we obtain

$$G^{(j,j)}(t+0,t) \geqq \sum_{k=1}^p |x_k^{(j)}(t)|^2/\lambda_k$$

on $a<t<b$. Since both sides are continuous, this holds on the closed interval I. Since the inequality holds for all p, $G^{(j,j)}(t+0,t)\geqq \sum_{k=1}^\infty |x_k^{(j)}(t)|^2/\lambda_k$.

THEOREM 2.4. *If $f\in D$ has Fourier coefficients a_k, the series $\sum_{k=1}^\infty a_k x_k^{(j)}(t)$ converges to $f^{(j)}(t)$ uniformly and absolutely on I for $j=0, 1, \cdots, n-1$.*

Proof.

$$\left[\sum_{k=q}^p |a_k x_k^{(j)}(t)|\right]^2 = \left[\sum_{k=q}^p (\lambda_k)^{1/2}|a_k| \cdot |x_k^{(j)}(t)|/(\lambda_k)^{1/2}\right]^2$$

$$\leqq \sum_{k=q}^p \lambda_k|a_k|^2 \sum_{k=q}^p |x_k^{(j)}(t)|^2/\lambda_k$$

by the Schwarz inequality. By Lemma 2.7, this is no greater than

$$\sum_{k=q}^p \lambda_k|a_k|^2 G^{(j,j)}(t+0,t).$$

The convergence of the series for $j=0, 1, \cdots, n-1$ now follows from the convergence of

$$\sum_{k=1}^\infty \lambda_k|a_k|^2.$$

Thus $f-\sum_{k=1}^\infty a_k x_k$ is a function of class C^{n-1} which satisfies the boundary conditions $Ux=0$, and thus belongs to D. Since $f-\sum_{k=1}^{\infty} a_k x_k$ is orthogonal to all eigenfunctions, by Theorem 2.3, $\sum_{k=1}^{\infty} a_k x_k$ converges uniformly to f. The series can be differentiated term by term $(n-1)$ times since the differentiated series converge.

THEOREM 2.5. *If $f\in H$ has Fourier coefficients a_k, then the Parseval equality, $\|f\|^2=\sum_{k=1}^{\infty}|a_k|^2$, is valid, so that $\sum_{k=1}^{\infty} a_k x_k$ converges to f in the norm of H.*

Proof. For $f\in D$, $f=\sum_{k=1}^{\infty} a_k x_k$, the convergence of the series being uniform. By Theorem 2.4, the series can be differentiated term by term and $Mf=\sum_{k=1}^{\infty} a_k Mx_k$, the convergence of this series also being uniform. We can multiply by $\bar{f}$ and integrate term by term, giving $\|f\|^2=\sum_{k=1}^{\infty} a_k(Mx_k, f)$ $=\sum_{k=1}^{\infty}|a_k|^2$. The result can be extended to the whole space H by a standard argument; see for example [1, Chapter 7, Theorem 4.2].

Let $\phi_j(t,\lambda)$ $[j=1,\cdots,n]$ be the solutions of $Lx=\lambda Mx$ which for some fixed c, $a<c<b$, satisfy $\phi_j^{(k-1)}(c,\lambda)=\delta_{jk}$ as before $(j, k=1,\cdots,n)$. The ϕ_j are linearly independent solutions and any eigenfunction x_k can be written in the form $x_k(t)=\sum_{j=1}^{n} r_{kj}\phi_j(t,\lambda_k)$, where the r_{kj} are complex constants. The Parseval equality becomes, for any $f\in H$, $\|f\|^2=\sum_{k=1}^{\infty}[f, x_k][x_k, f]$ $=\sum_{i,j=1}^{\infty} r_{ki}\bar{r}_{kj}[f,\phi_j][\phi_i, f]$. Here $[\phi_i, f]$ is not a true inner product, but merely a convenient designation for $\int_a^b M\phi_i(t)\bar{f}(t)dt$ and $[f,\phi_j]$ stands for the complex conjugate of $[\phi_j, f]$. Let $g_j(\lambda)=[f,\phi_j]$. Then $\|f\|^2=\sum_{k=1}^{\infty}\sum_{i,j=1}^{n} r_{ki}\bar{r}_{kj}\bar{g}_i(\lambda_k)g_j(\lambda_k)$. Let the function $\rho_{ij}(\lambda)$ be constant except at λ_k, $\rho_{ij}(0)=0$, $\rho_{ij}(\lambda+0)=\rho_{ij}(\lambda)$, and $\rho_{ij}(\lambda_k+0)-\rho_{ij}(\lambda_k-0)=\sum_p r_{pi}\bar{r}_{pj}$, the sum being over all p such that $\lambda_p=\lambda_k$. The matrix $\rho=(\rho_{ij})$ is hermitian; $\rho(\Delta)=\rho(\lambda)-\rho(\mu)$ is positive semidefinite if $\lambda>\mu$, $\Delta=(\mu,\lambda]$; and the total variation of ρ_{ij} is finite on every finite λ-interval. The matrix ρ is called the spectral matrix for the problem Lx $=\lambda Mx$, $Ux=0$. The Parseval equality now takes the form $\|f\|^2$ $=\int_{-\infty}^{\infty}\sum_{i,j=1}^{n}\bar{g}_i(\lambda)g_j(\lambda)d\rho_{ij}(\lambda)$. We can also write the expansion theorem in terms of the spectral matrix. By a straightforward argument we obtain $f(t)$ $=\int_{-\infty}^{\infty}\sum_{i,j=1}^{n}\phi_i(t,\lambda)g_j(\lambda)d\rho_{ij}(\lambda)$, where the integral converges to f in the norm of H. Note that these reformulations of the Parseval equality and the expansion theorem do not depend on the assumption $(Lu, u)>0$ for all $u\in D$ made after Lemma 2.5, but require only the original hypothesis (Lu, u) $\geqq K(u, u)$ for $u\in D$.

It will be convenient to denote functions such as G when considered as functions of t for fixed τ by $G(\ ,\tau)$, and when considered as functions of τ for fixed t by $G(t,\ )$.

THEOREM 2.6. *The spectral matrix ρ satisfies $2i\,\mathrm{Im}\,\lambda\int_{-\infty}^{\infty}(d\rho_{jk}(\sigma)/|\sigma-\lambda|^2)$ $=(\partial^{j+k-2}/\partial t^{j-1}\partial\tau^{k-1})H(c,c,\lambda)$, where $\mathrm{Im}\,\lambda\neq 0$; $j,k=1,\cdots,n$; and $H(t,\tau,\lambda)$ $=G(t,\tau,\lambda)-G(t,\tau,\bar{\lambda})$.*

Proof. If $\mathrm{Im}\,\lambda\neq 0$, $\mathrm{Im}\,\mu\neq 0$, let u be the solution of $Lu-\lambda Mu$ $=MG(\ ,\tau,\mu)$, $Uu=0$. Then $u(t)=\int_a^b G(t,s,\lambda)MG(s,\tau,\mu)ds$. Let $v=G(\ ,\tau,\lambda)$ $G(\ ,\tau,\mu)$, and then $v\in D$. Also, $Lv=LG(\ ,\tau,\lambda)-LG(\ ,\tau,\mu)$

$= \lambda MG(\ ,\tau,\lambda) - \mu MG(\ ,\tau,\mu) = \lambda Mv + (\lambda-\mu)MG(\ ,\tau,\mu)$, and thus $v = (\lambda-\mu)u$. Now $G(t,\tau,\lambda) - G(t,\tau,\mu) = (\lambda-\mu)\int_a^b G(t,s,\lambda)MG(s,\tau,\mu)ds$. We take $\mu = \bar{\lambda}$, and then $2i \operatorname{Im} \lambda \int_a^b G(t,s,\lambda)MG(s,\tau,\bar{\lambda})ds = H(t,\tau,\lambda)$. Since $G(s,\tau,\bar{\lambda}) = \overline{G}(\tau,s,\lambda)$, this yields $2i \operatorname{Im} \lambda \int_a^b G^{(j,0)}(t,s,\lambda)M\overline{G}^{(0,k)}(\tau,s,\lambda)ds = (\partial^{j+k}/\partial t^j \partial \tau^k)H(t,\tau,\lambda)$ for $j, k = 0, 1, \cdots, n-1$. From $Lx_p = \lambda M x_p + (\lambda_p - \lambda)Mx_p$, $Ux_p = 0$ it follows that $x_p(t) = (\lambda_p - \lambda)\int_a^b G(t,s,\lambda)Mx_p(s)ds$. Thus $\overline{G}^{(k,0)}(t,s,\lambda)$ has Fourier coefficients $\bar{x}_p^{(k)}(t)/(\lambda_p - \lambda)$, and by the Parseval equality,

$$
\begin{aligned}
2i \operatorname{Im} \lambda \sum_{p=1}^{\infty} x_p^{(j)}(t)\bar{x}_p^{(k)}(\tau)/|\lambda_p - \lambda|^2 \\
= 2i \operatorname{Im} \lambda \int_a^b G^{(j,0)}(t,s,\lambda)M\overline{G}^{(0,k)}(\tau,s,\lambda)\,ds = (\partial^{j+k}/\partial t^j \partial \tau^k)H(t,\tau,\lambda).
\end{aligned}
$$

Setting $t = \tau = c$, and using $\phi_j^{(k-1)}(c,\lambda) = \delta_{jk}$ and the definition of ρ_{jk}, we obtain the desired result.

If the coefficients of M are real, we can write

$$
Mx = \sum_{i=0}^{s} (d^i/dt^i)[g_{s-i}(t)(d^i x/dt^i)],
$$

with $m = 2s$ and g_{s-i} of class C^i on $a \leqq t \leqq b$. If M has this form, if $(-1)^i g_{s-i}(t) \geqq 0$, and if the boundary conditions $Ux = 0$ include the m conditions $x(a) = x'(a) = \cdots = x^{(s-1)}(a) = x(b) = x'(b) = \cdots = x^{(s-1)}(b) = 0$, then it is fairly easy to see that the space H is the set of all complex-valued functions f of class C^{s-1} on I such that $f^{(s-1)}$ is absolutely continuous on I, $f^{(s)} \in L^2(I)$, and $f(a) = \cdots = f^{(s-1)}(a) = f(b) = \cdots = f^{(s-1)}(b) = 0$. The norm of H is given in this case by $\|f\|^2 = \sum_{i=0}^{s}(-1)^i \int_a^b g_{s-i}(t)|(d^i f/dt^i)|^2 dt$.

3. The existence of the spectral matrix in the singular case. We now wish to consider the case where the coefficients of L or M have singular behavior at both ends of the interval $[a, b]$ or where $a = -\infty$, $b = \infty$. It suffices to examine the case of the infinite interval. For any finite subinterval $\delta = [\alpha, \beta]$ of I we assign boundary conditions $U_\delta x = 0$ such that the boundary value problem $Lx = Mx$, $U_\delta x = 0$ on δ is self-adjoint and the results of §2 are valid. We now assume that there exist real constants K and $d > 0$ such that if u belongs to any D_δ, $(Lu, u)_\delta \geqq K(u, u)_\delta$ and $(Mu, u)_\delta \geqq d(u, u)_\delta$. Here D_δ denotes the set of functions of class C^n on δ which satisfy $U_\delta x = 0$ and $(u, v)_\delta$ denotes the L^2-inner product on δ. The new inner product on δ will be denoted by $[u, v]_\delta$, the norm by $\|u\|_\delta$, and the corresponding Hilbert space by H_δ. For any such subinterval δ there exist real eigenvalues $\lambda_{\delta k} \geqq K/d$ and orthonormal eigenfunctions $x_{\delta k}(t)$, with $[x_{\delta j}, x_{\delta k}]_\delta = \delta_{jk}$. For any $f \in H_\delta$ we have the Parseval equality, $\|f\|_\delta^2 = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \bar{g}_{\delta i}(\lambda) g_{\delta j}(\lambda) d\rho_{\delta ij}(\lambda)$, and the expansion theorem $f(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \phi_i(t,\lambda) g_{\delta j}(\lambda) d\rho_{\delta ij}(\lambda)$, with equality in the norm of H_δ. Here $g_{\delta j} = [f, \phi_j]_\delta$, $x_{\delta k}(t) = \sum_{j=1}^{n} r_{\delta kj}\phi_j(t, \lambda_{\delta k})$, and $\rho_{\delta ij}$ depends on the interval δ.

We have shown the existence of a Green's function G_δ for the problem $Lx=\lambda Mx$, $U_\delta x=0$ on δ. We will prove the existence of a sequence of intervals δ_r tending to I such that G_{δ_r} converges. We have established a relation between G_δ and ρ_δ, and a limiting process will prove the existence of a spectral matrix ρ for the infinite interval. It is known that there exists a function $K(t, \tau)$ which serves as a kernel in the variation of constants formula for the solution of $Lx=f$. This function K is such that $\int_\delta K(t, \tau)f(\tau)d\tau$ is a solution of $Lx=f$ on any subinterval δ of I, and K has the same differentiability properties as any Green's function. In particular, $K^{(n-1,0)}$ has the same discontinuity at $t=\tau$ as $G^{(n-1,0)}$ if G is any Green's function, and $G-K$ is of class C^{n-1} as a function of t. We will choose $K(t, \tau)$ to be the function $K(t, \tau, 0)$ of Theorem 2.1, so that $K(t, \tau)=0$ for $t<\tau$.

Lemma 3.1. *The set of functions* $\{G_\delta\}$ *is uniformly bounded, and for* $n>1$, *equicontinuous on every compact* (t, τ, λ) *set where* $\operatorname{Im}\lambda\neq 0$. *If* $n=1$, $t=\tau$ *is expected.*

Proof. Let the closed interval δ_0 be contained properly in the closed interval δ_1, which in turn is contained properly in I. Let ζ be any real-valued function of class C^n on I such that $\zeta(t)=1$ for $t\in\delta_0$ and $\zeta(t)=0$ for $t\notin\delta_1$. Define the function $J(t, \tau)=\zeta(t)K(t, \tau)$ and let $\tau\in\delta_0$, $\delta_1\subset\delta$. Then $u=G_\delta(\ ,\tau,\lambda)-J(\ ,\tau)$ is of class C^n on δ except at $t=\tau$. In addition, u satisfies the boundary conditions $U_\delta u=0$, and $Lu-\lambda Mu=L_tG_\delta-L_tJ-\lambda M_tG_\delta+\lambda M_tJ=(\lambda M_t-L_t)J$, where the subscript t indicates that the operators L and M are applied to G_δ and J considered as functions of t. For convenience, we denote $\lambda M_tJ(t, s)-L_tJ(t, s)$, for each fixed $s\in\delta$, by $f(t)$. Then $f(t)$ is continuous except at $t=s$, and, since $J(t, s)=0$ for $t<s$ and $t\notin\delta_1$, $f(t)=0$ for $t<s$ and $t\notin\delta_1$. Now we can write $u(t)=G_\delta(t,\tau,\lambda)-J(t,\tau)=\int_\delta G_\delta(t,\tau,\lambda)f(\tau)d\tau=\int_{\delta_1}G_\delta(t,\tau,\lambda)f(\tau)d\tau$. If $\delta_1=[\alpha,\beta]$ this becomes $u(t)=\int_s^\beta G_\delta(t,\tau,\lambda)f(\tau)d\tau$ for each fixed $s\in\delta$. If $t<s$, we may differentiate under the integral sign $(n-1)$ times. Because of the discontinuity of $G_\delta^{(n-1,0)}$ at $t=\tau$, we obtain $u^{(n)}(t)=\int_s^\beta G_\delta^{(n,0)}(t,\tau,\lambda)f(\tau)d\tau+f(t)/p_0(t)$. It follows easily that $Lu=\lambda Mu+f$ for $t<s$. At $t=s$, $u^{(n-1)}$ is continuous since $u^{(n-1)}(t)=\int_s^\beta G_\delta^{(n-1,0)}(t,\tau,\lambda)f(\tau)d\tau$ regardless of whether $t<s$ or $t>s$. Also, $u^{(n)}(t)$ has a simple discontinuity of magnitude $f(t)/p_0(t)$ at $t=s$. Since u is of class C^{n-1} and L is assumed formally self-adjoint, it is easily seen, using the fact that $U_\delta u=0$, that $(Lu, u)_\delta=(u, Lu)_\delta$, which yields $2i\operatorname{Im}\lambda(Mu, u)_\delta=2i\operatorname{Im}(u, f)_\delta$. We now use $|||y|||_\delta$ to denote the usual L^2-norm of any function y on δ. The Schwarz inequality gives $|\operatorname{Im}\lambda|(Mu, u)_\delta=|\operatorname{Im}(u, f)_\delta|\leqq|(u, f)_\delta|\leqq|||u|||_\delta|||f|||_{\delta_1}$. Since $(Mu, u)_\delta\geqq d|||u|||_\delta^2$, $|\operatorname{Im}\lambda|\cdot|||u|||_\delta\leqq d^{-1}|||f|||_{\delta_1}$. Applying this inequality to $u=G_\delta(\ ,\tau,\lambda)-J(\ ,\tau)$ for $\tau\in\delta_0$, $|||G_\delta(\ ,\tau,\lambda)|||_\delta\leqq|||J(\ ,\tau)|||_{\delta_1}+(1/d|\operatorname{Im}\lambda|)|||f|||_{\delta_1}$. In addition, $G_\delta(t,\tau,\lambda)=J(t,\tau)+\int_{\delta_1}G_\delta(t,s,\lambda)f(s)ds$ yields $|G_\delta(t,\tau,\lambda)|\leqq|J(t,\tau)|+|||G_\delta(t,\ ,\lambda)|||_\delta|||f|||_{\delta_1}$. Thus $\{G_\delta\}$ is uniformly bounded for $\tau\in\delta_0$, $\delta\supset\delta_1$ and λ in any compact set S in the λ-plane where $\operatorname{Im}\lambda\neq 0$. The symmetry relation $G_\delta(t, \tau, \lambda) = \bar{G}_\delta(\tau, t, \bar{\lambda})$ shows that $\{G_\delta\}$ is uniformly bounded for $t, \tau \in \delta_0$, $\delta \supset \delta_1$, $\lambda \in S$, provided S is chosen symmetric about the real axis in the λ-plane. For $n > 1$,

$$G_\delta^{(0,1)}(t, \tau, \lambda) = J^{(0,1)}(t, \tau) + \int_{\delta_1} G_\delta(t, s, \lambda)(\partial/\partial\tau)[\lambda M_s J(s, \tau) - L_s J(s, \tau)]ds,$$

and the Schwarz inequality gives the uniform boundedness of the set $\{G_\delta^{(0,1)}\}$ for $t, \tau \in \delta_0$, $\lambda \in S$. The symmetry relation ensures the uniform boundedness of $\{G_\delta^{(0,1)}\}$. Since $\{G_\delta\}$ is uniformly bounded and analytic in λ, $\{(\partial G_\delta/\partial\lambda)\}$ is uniformly bounded. The uniform boundedness of all first order partial derivatives implies the equicontinuity of the set $\{G_\delta\}$.

This lemma, together with Ascoli's theorem, proves that there exists a sequence of intervals δ_r, contained in I and tending to I as $r \to \infty$ such that the corresponding Green's functions $G_r = G_{\delta_r}$ converge uniformly on a fixed compact subset Γ_1 of $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$, which is symmetric about the real axis in the λ-plane. A subsequence will tend to a limit function uniformly on a compact subset $\Gamma_2 \supset \Gamma_1$. By taking a sequence $\{\Gamma_i\}$ tending to $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$, and using the diagonal process, we see that there exists a sequence of Green's functions G_r which tend uniformly on any compact subset of $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$ to a limit function G defined for $t, \tau \in I$, $\operatorname{Im} \lambda \neq 0$. Because of the uniform convergence, G is continuous in (t, τ, λ) and analytic in λ. The relation $G_r(t, \tau, \lambda) = \bar{G}_r(\tau, t, \bar{\lambda})$ implies $G(t, \tau, \lambda) = \bar{G}(\tau, t, \bar{\lambda})$.

THEOREM 3.1. *Let G be the limit of any convergent sequence of Green's functions G_r of the set $\{G_\delta\}$. Then G is continuous in $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$ ($t \neq \tau$ if $n = 1$), analytic in λ and has the following properties*:

(i) *$G^{(k,0)}$ exist and are continuous in $a < t, \tau < b$, $\operatorname{Im} \lambda \neq 0$ for $k = 0, 1, \cdots, n-2$. $G^{(n-1,0)}$ and $G^{(n,0)}$ are continuous in $t < \tau$ and $t \geqq \tau$.*

(ii) *$G^{(n-1,0)}(\tau+0, \tau, \lambda) - G^{(n-1,0)}(\tau-0, \tau, \lambda) = 1/p_0(\tau)$ for $a < \tau < b$.*

(iii) *As a function of t, G satisfies $Lx = \lambda Mx$ if $t \neq \tau$.*

(iv) *$G(t, \tau, \lambda) = \bar{G}(\tau, t, \bar{\lambda})$.*

(v) *$G(t, \ , \lambda)$ belongs to the class $L^2(I)$.*

(vi) *If $f \in L^2(I)$, the function v defined by $v(t) = \int_a^b G(t, \tau, \lambda) f(\tau) d\tau$ belongs to $L^2(I)$ and $Lv = \lambda Mv + f$.*

Proof. The representation

$$G_\delta(t, \tau, \lambda) = J(t, \tau) + \int_{\delta_1} G_\delta(t, s, \lambda)[\lambda M_s J(s, \tau) - L_s J(s, \tau)]ds$$

may be differentiated with respect to τ $n-1$ times under the integral sign. Taking $\delta = \delta_r$ and letting $r \to \infty$, we obtain similar formulae for G instead of G_δ. This proves the existence of the partial derivatives $G^{(0,k)}$ and the relation (iv), which has already been proved, proves the existence of the partial derivatives $G^{(k,0)}$. It is also clear that $G^{(n-1,0)}$ has the same discontinuity at $t=\tau$ as $K^{(n-1,0)}$, which proves (ii). It follows from the representation $G(t, \tau, \lambda)=K(t, \tau)+\int_{\delta_1}G(t, s, \lambda)[\lambda M_s - L_s]J(s, \tau)ds$ that as a function of t, G satisfies $Lx=\lambda Mx$ for $t\neq\tau$, which proves (iii). To prove (v), we use $|||G_\delta(\ , \tau, \lambda)|||_\delta \leqq |||J(\ , \tau)|||_{\delta_1}+(1/d|\operatorname{Im}\lambda|)|||f|||_{\delta_1}$, where f is defined as in Lemma 3.1. There exists a constant A depending only on δ_0 and δ_1 such that $|||G_\delta(\ , \tau, \lambda)|||_\delta \leqq (A/|\operatorname{Im}\lambda|)(|\lambda|+1)+A$. For $\tilde{\delta}\subset\delta$, $|||G_\delta(\ , \tau, \lambda)|||_{\tilde{\delta}} \leqq |||G_\delta(\ , \tau, \lambda)|||_\delta$, and letting $\delta\rightarrow I$ via the sequence δ_r and then $\tilde{\delta}\rightarrow I$, we see that $G(\ , \tau, \lambda)\in L^2(I)$ for any fixed (τ, λ), $\operatorname{Im}\lambda\neq 0$. For fixed (t, λ), $\operatorname{Im}\lambda\neq 0$, $G(t, \ , \lambda)\in L^2(I)$ by the symmetry relation (iv), and (v) is proved. Finally, if $f\in L^2(I)$, then by (v) the integral $\int_a^b G(t, \tau, \lambda)f(\tau)d\tau$ $[a<t<b, \operatorname{Im}\lambda\neq 0]$ converges absolutely and uniformly in t for any compact subinterval of I. It defines a function $v(t)$, and in exactly the same manner as in [1, p. 277] it may easily be verified that $v\in C^{n-1}$, $v^{(n-1)}$ is absolutely continuous, and $Lv=\lambda Mv+f$. If $f\in L^2(I)$ and $u(t)=\int_\delta G_\delta(t, \tau, \lambda)f(\tau)d\tau$, then $Lu-\lambda Mu=f$, $U_\delta u=0$. The self-adjointness condition $(Lu, u)_\delta=(u, Lu)_\delta$ gives $(\lambda Mu+f, u)_\delta =(u, \lambda Mu+f)_\delta$, which gives $2i \operatorname{Im}\lambda(Mu, u)_\delta=\int_\delta(u\bar{f}-f\bar{u})dt$. Then $d|\operatorname{Im}\lambda| \cdot|||u|||_\delta^2 \leqq |\operatorname{Im}\lambda|(Mu,u)_\delta \leqq |(u,f)_\delta| \leqq |||u|||_\delta|||f|||_\delta$, and thus $|||u|||_\delta \leqq(1/d|\operatorname{Im}\lambda|)|||f|||_\delta$. Letting $\delta=\delta_r$, $r\rightarrow\infty$, we obtain $|||v|||\leqq(1/d|\operatorname{Im}\lambda|)|||f|||$, which proves $v\in L^2(I)$.

If G is the limit of any convergent sequence $\{G_r\}$ of the set $\{G_\delta\}$, we define $H(t, \tau, \lambda)=G(t, \tau, \lambda)-G(t, \tau, \bar{\lambda})$, $P_{\delta jk}(\lambda)=H_\delta^{(j-1,k-1)}(c, c, \lambda)$, and $P_{jk}(\lambda)=H^{(j-1,k-1)}(c, c, \lambda)$ for $j, k=1, \cdots, n$.

THEOREM 3.2. *Let $\{G_r\}$ be any convergent sequence of the set $\{G_\delta\}$ and let ρ_r be the spectral matrix associated with G_r. Then there exists a hermitian nondecreasing matrix ρ, whose elements are of bounded variation on every finite λ-interval, such that $\rho_r(\lambda_2)-\rho_r(\lambda_1)\rightarrow\rho(\lambda_2)-\rho(\lambda_1)$ as $r\rightarrow\infty$ if ρ is continuous at λ_1 and λ_2. At such λ_1 and λ_2, $\rho_{jk}(\lambda_2)-\rho_{jk}(\lambda_1)=(1/2\pi i)\lim_{\epsilon\to 0+}\int_{\lambda_1}^{\lambda_2}P_{jk}(v+i\epsilon)dv$.*

Proof. By Theorem 2.6, $2i \operatorname{Im}\lambda\int_{-\infty}^{\infty}(d\rho_{rjk}(\sigma)/|\sigma-\lambda|^2)=P_{rjk}(\lambda)$. Taking $\lambda=i$ in this relation and using the fact that ρ_{rjj} is monotone it follows that $\int_{-\eta}^{\eta}(d\rho_{rjj}(\sigma)/(1+\sigma^2))\leqq P_{rjj}(i)/2i$. It is easy to see that $H_r^{(j-1,k-1)}(t, \tau, \lambda)$ is continuous for $j, k=1, \cdots, n$; $\operatorname{Im}\lambda\neq 0$ and that $H_r^{(j-1,k-1)}$ tends uniformly to $H^{(j-1,k-1)}$ on any compact (t, τ, λ) set where $\operatorname{Im}\lambda\neq 0$ for $j, k=1, \cdots, n$. It follows that $P_{rjj}(i)\rightarrow P_{jj}(i)$, so that $P_{rjj}(i)/2i$ is bounded and $\int_{-\infty}^{\infty}(d\rho_{rjj}(\sigma)/(1+\sigma^2))<A$ for some constant A and $j=1, \cdots, n$. Thus $|\rho_{rjj}(\sigma)|<A(1+\sigma^2)$, and if $\Delta=(\lambda_1, \lambda_2]$, $\rho_{rjk}(\Delta)=\rho_{rjk}(\lambda_2)-\rho_{rjk}(\lambda_1)$, the inequality $|\rho_{rjk}(\Delta)|^2\leqq\rho_{rjj}(\Delta)\rho_{rkk}(\Delta)$ shows that the total variation of ρ_{rjk} on any finite λ-interval is bounded independent of r. The Helly selection theorem implies the existence of a subsequence of $\{\rho_r\}$ tending to a limiting matrix ρ. Since each ρ_r is hermitian, nondecreasing, and has bounded variation on every finite λ-interval, the limiting matrix ρ also has these properties. If $\operatorname{Im}\lambda\neq 0$, $\operatorname{Im}\lambda_0\neq 0$, we consider $\int_{-\infty}^{\infty}(1/|\sigma-\lambda|^2-1/|\sigma-\lambda_0|^2)d\rho_{rjk}(\sigma)$ over the intervals $(-\infty, -\mu)$, $(-\mu, \mu)$ and (μ, ∞). If we let $r\rightarrow\infty$ and then $\mu\rightarrow\infty$, this tends

to the same expression with ρ_{rjk} replaced by ρ_{jk}. But the original integral has the value $P_{rjk}(\lambda)/2i \operatorname{Im} \lambda - P_{rjk}(\lambda_0)/2i \operatorname{Im} \lambda_0$, which tends to $P_{jk}(\lambda)/2i \operatorname{Im} \lambda - P_{jk}(\lambda_0)/2i \operatorname{Im} \lambda_0$ as $r \to \infty$. Thus $P_{jk}(\lambda) = 2i \operatorname{Im} \lambda [\int_{-\infty}^{\infty} (d\rho_{jk}(\sigma)/|\sigma - \lambda|^2) + h]$, where h is a constant independent of λ, for $\operatorname{Im} \lambda \neq 0$. Now, if ρ is continuous at λ_1 and λ_2, $\lim_{\epsilon \to 0+} \int_{\lambda_1}^{\lambda_2} P_{jk}(v + i\epsilon) dv = \lim_{\epsilon \to 0+} \int_{\lambda_1}^{\lambda_2} \int_{-\infty}^{\infty} (2i\epsilon d\rho_{jk}(\sigma)/((\sigma - v)^2 + \epsilon^2)) dv = 2\pi i [\rho_{jk}(\lambda_2) - \rho_{jk}(\lambda_1)]$, interchanging the order of integration. Since $P_{rjk} \to P_{jk}$, it follows that $\rho_r(\lambda_2) - \rho_r(\lambda_1) \to \rho(\lambda_2) - \rho(\lambda_1)$, and the theorem is proved.

If it is assumed that the equation $Lx = \lambda Mx$ has no nontrivial solutions in $L^2(I)$ for any λ with $\operatorname{Im} \lambda \neq 0$, the Green's function of Theorem 3.1 is unique since the difference between any two functions with the properties (i)–(vi) of Theorem 3.1 is a solution of $Lx = \lambda Mx$ which is in $L^2(I)$, and which therefore must vanish identically. In view of Theorem 3.2, any condition which implies the uniqueness of the Green's function also implies the uniqueness of the spectral matrix ρ.

4. The Parseval equality and the expansion theorem in the singular case. Let D denote the set of functions of class C^n which vanish outside a finite subinterval of I, and let H be the Hilbert space completion of D in the inner product $[f, g] = (Mf, g) = \int_I Mf(t)\bar{g}(t) dt$. Then H is a Hilbert space with inner product $[f, g]$ and norm $\|f\| = [f, f]^{1/2}$. For any limiting matrix ρ, define the space $L^2(\rho)$ to be the set of all vector functions $g = (g_j)$ $[j = 1, \cdots, n]$ measurable with respect to ρ and such that $\|g\|^2 = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \bar{g}_i(\lambda) g_j(\lambda) d\rho_{ij}(\lambda)$ is finite.

THEOREM 4.1. *Let ρ be any limiting matrix. If $f \in H$, there exists $g \in L^2(\rho)$ such that if $g_{\delta j} = [f, \phi_j]_\delta$, then $\|g - g_\delta\| \to 0$ as $\delta \to I$. In terms of this g, the Parseval equality $\|f\| = \|g\|$ is valid.*

Proof. First consider a function $f \in D$. If $\delta = [\alpha, \beta]$ and α, β are near enough to a, b respectively, then $f \in D_\delta$ and $\|f\|^2 = \|f\|_\delta^2 = \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \bar{g}_{\delta i}(\lambda) g_{\delta j}(\lambda) d\rho_{\delta ij}(\lambda)$. We know that $\lambda_{\delta k} \geqq K/d$ for all δ, and if $\Lambda > -K/d$, $L_1 u = Lu + \Lambda Mu$, then $(L_1 u, u)_\delta > 0$ for all $u \in D_\delta$. Thus $(L_1 f, f) = (L_1 f, f)_\delta \geqq \sum_{k=1}^{\infty} (\lambda_{\delta k} + \Lambda) |a_{\delta k}|^2 = \int_0^{\infty} \sum_{i,j=1}^{n} (\lambda + \Lambda) \bar{g}_{\delta i}(\lambda) g_{\delta j}(\lambda) d\rho_{\delta ij}(\lambda)$. We take $g_i = g_{\delta i}$, and then for any $A > 0$, $(L_1 f, f)/(\Lambda + A) \geqq \int_A^{\infty} \sum_{i,j=1}^{n} (\lambda + \Lambda/(A + \Lambda)) \bar{g}_i(\lambda) g_j(\lambda) d\rho_{\delta ij}(\lambda) \geqq \int_A^{\infty} \sum_{i,j=1}^{n} \bar{g}_i(\lambda) g_j(\lambda) d\rho_{\delta ij}(\lambda)$. Thus $\left| \|f\|^2 - \int_{-A}^{A} \sum_{i,j=1}^{n} \bar{g}_i(\lambda) g_j(\lambda) d\rho_{\delta ij}(\lambda) \right| \leqq (L_1 f, f)/(\Lambda + A)$. Letting $\delta \to I$ through the sequence δ_r, we obtain the same inequality with $\rho_{\delta ij}$ replaced by ρ_{ij}, and then letting $A \to \infty$, we see that $g \in L^2(\rho)$ and $\|f\| = \|g\|$. Since D is dense in H, for any $f \in H$ there exists a sequence $f^{(k)}$ in D such that $\lim_{k \to \infty} \|f - f^{(k)}\| = 0$. If $g^{(k)}$ is the (vector) transform of $f^{(k)}$, the Parseval equality applied to $f^{(k)} - f^{(p)}$ gives $\|f^{(k)} - f^{(p)}\| = \|g^{(k)} - g^{(p)}\|$. Since the left side tends to zero as $k, p \to \infty$, $g^{(k)}$ is a Cauchy sequence in $L^2(\rho)$, and since $L^2(\rho)$ is complete, there exists $g \in L^2(\rho)$ such that $\lim_{k \to \infty} \|g - g^{(k)}\| = 0$. We now have $\|f\| = \lim_{k \to \infty} \|f^{(k)}\| = \lim_{k \to \infty} \|g^{(k)}\| = \|g\|$, which gives the Parseval equality for any $f \in H$.

The proof of the expansion theorem requires some results about the operator M obtained by Friedrichs in [2]. The Hilbert space H is a subset of $L^2(I)$ and M, considered as an operator on $L^2(I)$, has a self-adjoint extension (also denoted by M) whose domain D_M is contained in H and whose range is dense in $L^2(I)$.

LEMMA 4.1. *If $f \in H$ and $P \in D_M$, then $[P, f] = \int_I MP(t)\bar{f}(t)dt$.*

Proof. There exists a sequence f_j in D_M such that $\|f-f_j\| \to 0$. Since $\|f-f_j\|^2 \geqq d|||f-f_j|||^2$, $|||f-f_j||| \to 0$. There exists a subsequence, also denoted by f_j, which converges almost everywhere to f. Since f, f_j, and MP all belong to $L^2(I)$, $MP\bar{f}$ and $MP\bar{f}_j$ belong to $L^1(I)$. Now $MP\bar{f}_j \to MP\bar{f}$ almost everywhere, and $\int_I MP(t)\bar{f}_j(t)dt \to \int_I MP(t)\bar{f}(t)dt$. By the continuity of the inner product, $[P, f] = \lim_{j\to\infty} [P, f_j] = \lim_{j\to\infty} \int_I MP(t)\bar{f}_j(t)dt = \int_I MP(t)\bar{f}(t)dt$.

THEOREM 4.2. *Let ρ be any limiting matrix, and let $f \in H$. If g is the transform of f, whose existence is assured by Theorem* 4.1, *then the expansion $f(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^n \phi_i(t, \lambda)g_j(\lambda)d\rho_{ij}(\lambda)$ is valid, where the integral converges to f in the norm of H, possibly after redefinition on a null set.*

Proof. Let $\Delta = (\mu, \nu]$ and define $f_\Delta(t) = \int_\Delta \sum_{i,j=1}^n \phi_i(t, \lambda)g_j(\lambda)d\rho_{ij}(\lambda)$. We will show that after redefinition of f_Δ on a null set, $f - f_\Delta \in H$ and $\|f-f_\Delta\| \to 0$ as $\Delta \to (-\infty, \infty)$. If $f^{(1)}, f^{(2)} \in H$ have transforms $g^{(1)}, g^{(2)}$ respectively, then application of the Parseval equality to $f^{(1)} + i^p f^{(2)}$ $[p = 0, 1, 2, 3]$ gives $[f^{(1)}, f^{(2)}] = \int_{-\infty}^{\infty} \sum_{i,j=1}^n \bar{g}_i^{(2)}(\lambda)g_j^{(1)}(\lambda)d\rho_{ij}(\lambda)$. Let $P \in D_M$ and let Q be the transform of P. Then

$$\int_I MP(t)\bar{f}_\Delta(t)dt = \int_\Delta \sum_{i,j=1}^n \bar{g}_j(\lambda)\left\{\int_I \bar{\phi}_i(t, \lambda)MP(t)dt\right\} d\bar{\rho}_{ij}(\lambda)$$
$$= \int_\Delta \sum_{i,j=1}^n \bar{g}_j(\lambda)Q_j(\lambda)d\bar{\rho}_{ij}(\lambda).$$

Also, $[P, f] = \int_I MP(t)\bar{f}(t)dt = \int_{-\infty}^{\infty} \sum_{i,j=1}^n \bar{g}_j(\lambda)Q_i(\lambda)d\bar{\rho}_{ij}(\lambda)$, using Lemma 4.1 and the Hermitian nature of ρ. Subtracting and letting $\Delta^c = (-\infty, \infty) - \Delta$, we obtain $\int_I MP(\bar{f} - \bar{f}_\Delta)dt = \int_{\Delta^c} \sum_{i,j=1}^n \bar{g}_j(\lambda)Q_i(\lambda)d\bar{\rho}_{ij}(\lambda)$. Using the Schwarz inequality, we find

$$\left|\int_I MP(\bar{f} - \bar{f}_\Delta)dt\right|^2 \leqq \int_{\Delta^c} \sum_{i,j=1}^n \bar{g}_j(\lambda)g_i(\lambda)d\bar{\rho}_{ij}(\lambda) \int_{\Delta^c} \sum_{i,j=1}^n \bar{Q}_j(\lambda)Q_i(\lambda)d\bar{\rho}_{ij}(\lambda)$$
$$\leqq \|P\|^2 \int_{\Delta^c} \sum_{i,j=1}^n \bar{g}_j(\lambda)g_i(\lambda)d\bar{\rho}_{ij}(\lambda).$$

By choosing Δ sufficiently large, we can make $\left|\int_I MP(\bar{f} - \bar{f}_\Delta)dt\right| \leqq \epsilon\|P\|$ for any $\epsilon > 0$. The bounded linear functional $\int_I MP(\bar{f} - \bar{f}_\Delta)dt$ defined for P in the dense subset D_M of H can be extended to H without increasing its norm, and this functional can be written as $[P, h_\Delta]$ for all $P \in H$ and some $h_\Delta \in H$

with $\|h_\Delta\| \leqq \epsilon$. For $P \in D_M$, $\int_I MP(\bar{f}-\bar{f}_\Delta)dt = [P, h_\Delta] = \int_I MP\bar{h}_\Delta dt$, by Lemma 4.1, and $\int_I MP(\bar{f}-\bar{f}_\Delta-\bar{h}_\Delta)dt=0$. Since the set of functions MP for $P \in D_M$ is dense in $L^2(I)$, it follows that $f-f_\Delta-h_\Delta$ belongs to $L^2(I)$ and is zero almost everywhere. Redefining f_Δ on a null set if necessary, we have $f-f_\Delta=h_\Delta \in H$, and $\|f-f_\Delta\| \leqq \epsilon$, as desired.

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